SELF-ADJOINT COMMUTING DIFFERENTIAL OPERATORS AND COMMUTATIVE SUBALGEBRAS OF THE WEYL ALGEBRA

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Abstract. In this paper we study self-adjoint commuting ordinary differential operators. We find sufficient conditions when an operator of fourth order commuting with an operator of order $4g+2$ is self-adjoint. We introduce an equation on potentials $V(x), W(x)$ of the self-adjoint operator $L=(\partial_x^2+V)^2+W$ and some additional data. With the help of this equation we find the first example of commuting differential operators of rank two corresponding to a spectral curve of arbitrary genus. These operators have polynomial coefficients and define commutative subalgebras of the first Weyl algebra.

1. Introduction

The problem of finding commuting differential operators is a classical problem of differential equations (for the first results see [1]–[3]). In the case of operators of rank greater than one, this problem has not been solved until now. In this paper we study self-adjoint commuting ordinary differential operators. One of the main results of this paper is the following. We find an example of commuting differential operators of rank two corresponding to spectral curves of arbitrary genus.

If two differential operators

$$L_n=\partial_x^n+\sum_{i=0}^{n-2}u_i(x)\partial_x^i,\quad L_m=\partial_x^m+\sum_{i=0}^{m-2}v_i(x)\partial_x^i$$

commute, then there is a nonzero polynomial $R(z,w)$ such that $R(L_n,L_m)=0$ (see [3]). The curve Γ defined by $R(z,w)=0$ is called the *spectral curve.* This curve parametrizes common eigenvalues of the operators. If

$$L_n\psi=z\psi,\qquad L_m\psi=w\psi,$$

then $(z,w)\in\Gamma$. For almost all $(z,w)\in\Gamma$ the dimension of the space of common eigenfunctions ψ is the same. The dimension is called the *rank*. The rank equals the greatest common divisor of m and n.

In this paper we consider only commuting ordinary differential operators whose spectral curves are smooth. Commutative rings of such operators were classified by

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Krichever [4], [5]. The ring is determined by the spectral curve and some additional spectral data. If the rank is one, then the spectral data define commuting operators by explicit formulas (see [4]). In the case of operators of rank greater than one there are the following results. Krichever and Novikov [6], [7] using the method of deformation of Tyurin parameters found operators of rank two corresponding to an elliptic spectral curves. These operators were studied in the papers [8]–[16]. Mokhov [17], using the same method found operators of rank three also corresponding to elliptic spectral curves. Besides this there are examples of operators of rank grater than one corresponding to spectral curves of genus $2, 3$ and 4 (see [18]–[21]).

The main results of this paper are the following. We consider a pair L_4, L_{4g+2} of commuting differential operators of rank two whose spectral curve is a hyperelliptic curve Γ of genus g

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \cdots + c_0. \tag{1}$$

Operators L_4 and L_{4g+2} satisfy the equation $(L_{4g+2})^2 = F_g(L_4)$. The curve Γ has a holomorphic involution

$$\sigma : \Gamma \to \Gamma, \qquad \sigma(z, w) = (z, -w).$$

Common eigenfunctions of L_4 and L_{4g+2} satisfy the second order differential equation [5]

$$\psi''(x, P) = \chi_1(x, P)\psi'(x, P) + \chi_0(x, P)\psi(x, P). \tag{2}$$

The coefficients $\chi_0(x, P), \chi_1(x, P)$ are rational functions on Γ with $2g$ simple poles depending on x, χ_0 has also an additional simple pole at infinity. These functions satisfy Krichever's equations (see below). To find operators L_4, L_{4g+2} it is enough to find χ_0, χ_1.

It is not difficult to prove that if χ_1 is invariant under the involution σ, then the operator L_4 is self-adjoint. S.P. Novikov has proposed the conjecture that the inverse is also true. In this paper we prove this conjecture.

Theorem 1 *The operator L_4 is self-adjoint if and only if*

$$\chi_1(x, P) = \chi_1(x, \sigma(P)). \tag{3}$$

At $g = 1$ Theorem 1 was proved by Grinevich and Novikov [8].

Let us assume that the operator L_4 is self-adjoint

$$L_4 = (\partial_x^2 + V(x))^2 + W(x),$$

then the functions χ_0, χ_1 have simple poles at some points

$$(\gamma_i(x), \pm\sqrt{F_g(\gamma_i(x))}),\ 1 \le i \le g.$$

In the next theorem we find the form of $\chi_0(x, P), \chi_1(x, P)$.

Theorem 2 *If operator L_4 is self-adjoint, then*

$$\chi_0 = -\frac{1}{2}\frac{Q''}{Q} + \frac{w}{Q} - V, \qquad \chi_1 = \frac{Q'}{Q},$$

where $Q = (z - \gamma_1(x)) \dots (z - \gamma_g(x))$. Functions Q, V, W satisfy the equation

$$4F_g(z) = 4(z-W)Q^2 - 4V(Q')^2 + (Q'')^2 - 2Q'Q^{(3)} + 2Q(2V'Q' + 4VQ'' + Q^{(4)}), \tag{4}$$

where $Q', Q'', Q^{(k)}$ *mean* $\partial_x Q, \partial_x^2 Q, \partial_x^k Q$.

To find self-adjoint operators L_4, L_{4g+2} it is enough to solve the equation (4).

In this paper we find partial solutions of the equation for arbitrary g. These solutions correspond to operators with polynomial coefficients.

Theorem 3 *The operator*

$$L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x, \qquad \alpha_3 \neq 0$$

commutes with a differential operator $L_{4g+2}^\sharp$ *of order* $4g+2$. *The operators* $L_4^\sharp$, $L_{4g+2}^\sharp$ *are operators of rank two. For generic values of parameters* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ *the spectral curve is a nonsingular hyperelliptic curve of genus* g.

If $g = 1$, $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, then the operators $L_4^\sharp, L_{4g+2}^\sharp$ coincide with the famous Dixmier operators [22] whose spectral curve is an elliptic curve. Operators $L_4^\sharp, L_{4g+2}^\sharp$ define commutative subalgebras in the first Weyl algebra A_1. Theorem 3 means that the equation $Y^2 = X^{2g+1} + c_{2g}X^{2g} + \cdots + c_0$ has nonconstant solutions $X, Y \in A_1$ for some c_i. It is easy to see that the group $Aut(A_1)$ preserves the space of all such solutions. It would be very interesting to describe the orbits of $Aut(A_1)$ in the space of solutions under the action of $Aut(A_1)$. This gives a chance to compare $End(A_1)$ and $Aut(A_1)$ (the Dixmier conjecture is: $End(A_1) = Aut(A_1)$).

In Section 2 we recall the method of deformations of Tyurin parameters. In Sections 3–5 we prove Theorems 1–3.

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2. Operators of rank $l > 1$

Common eigenfunctions of commuting differential operators are Baker–Akhiezer functions. Let me recall the definition of the Baker–Akhiezer function at $l > 1$ [5]. We take the *spectral data*

$$\{\Gamma, q, k^{-1}, \gamma, v, \omega(x)\},$$

where Γ is a Riemann surface of genus g, q is a fixed point on Γ, k^{-1} is a local parameter near q,

$$\omega(x) = (\omega_0(x), \dots, \omega_{l-2}(x))$$

is a set of smooth functions, $\gamma = \gamma_1 + \cdots + \gamma_{lg}$ is a divisor on Γ, v is a set of vectors

$$v_1, \dots, v_{lg}, \qquad v_i = (v_{i,1}, \dots, v_{i,l-1}).$$

The pair (γ, v) is called the *Tyurin parameters*. The Tyurin parameters define a stable holomorphic vector bundle on Γ of rank l and degree lg with holomorphic sections $\eta_1, \dots, \eta_l$. The points $\gamma_1, \dots, \gamma_{lg}$ are the points of the linear dependence

$$\eta_l(\gamma_i) = \sum_{i=1}^{l-1} v_{j,i}\eta_j(\gamma_i).$$

The vector-function $\psi = (\psi_1, \dots, \psi_l)$ is defined by the following properties.

1. In the neighbourhood of q the vector-function ψ has the form

$$\psi(x, P) = \left(\sum_{s=0}^{\infty} \xi_s(x) k^{-s}\right) \Psi_0(x, k),$$

where $\xi_0=(1,0,\dots,0), \xi_i(x)=(\xi_i^1(x),\dots,\xi_i^l(x))$, the matrix Ψ_0 satisfies the equation

$$\frac{d\Psi_0}{dx}=A\Psi_0,\ A=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 0 & 1\\ k+\omega_0 & \omega_1 & \omega_2 & \dots & \omega_{l-2} & 0 \end{pmatrix}.$$

2. The components of ψ are meromorphic functions on $\Gamma\backslash\{q\}$ with the simple poles $\gamma_1,\dots,\gamma_{lg}$, and

$$\mathrm{Res}_{\gamma_i}\psi_j=v_{i,j}\mathrm{Res}_{\gamma_i}\psi_l, \quad 1\le i\le lg,\ 1\le j\le l-1.$$

For the rational function $f(P)$ on Γ with the unique pole of order n at q there is a linear differential operator $L(f)$ of order ln such that

$$L(f)\psi(x,P)=f(P)\psi(x,P).$$

For two such functions $f(P),g(P)$ operators $L(f)$, $L(g)$ commute.

The main difficulty to construct operators of rank $l>1$ is the fact that the Baker–Akhiezer function is not found explicitly. But the operators can be found by the method of deformation of Tyurin parameters.

The common eigenfunctions of commuting differential operators of rank l satisfy the linear differential equation of order l

$$\psi^{(l)}(x,P)=\chi_0(x,P)\psi(x,P)+\dots+\chi_{l-1}(x,P)\psi^{(l-1)}(x,P).$$

Coefficients χ_i are rational functions on Γ [5] with simple poles $P_1(x),\dots,P_{lg}(x)\in\Gamma$, and with the following expansions in the neighbourhood of q

$$\chi_0(x,P)=k+g_0(x)+O(k^{-1}),$$

$$\chi_j(x,P)=g_j(x)+O(k^{-1}),\quad 0<j<l-1,$$

$$\chi_{l-1}(x,P)=O(k^{-1}).$$

Let $k-\gamma_i(x)$ be a local parameter near $P_i(x)$. Then

$$\chi_j=\frac{c_{i,j}(x)}{k-\gamma_i(x)}+d_{i,j}(x)+O(k-\gamma_i(x)).$$

Functions $c_{ij}(x),d_{ij}(x)$ satisfy the following equations [5].

Theorem 4

$$c_{i,l-1}(x)=-\gamma_i'(x), \tag{5}$$

$$d_{i,0}(x)=v_{i,0}(x)v_{i,l-2}(x)+v_{i,0}(x)d_{i,l-1}(x)-v_{i,0}'(x), \tag{6}$$

$$d_{i,j}(x)=v_{i,j}(x)v_{i,l-2}(x)-v_{i,j-1}(x)+v_{i,j}(x)d_{i,l-1}(x)-v_{i,j}'(x), j\ge 1, \tag{7}$$

where

$$v_{i,j}(x)=\frac{c_{i,j}(x)}{c_{i,l-1}(x)},\quad 0\le j\le l-1,\ 1\le i\le lg.$$

To find χ_i one should solve equations (5)–(7).

3. Proof of Theorem 1

In the case of operators of rank two the common eigenfunctions of L_4 and L_{4g+2} satisfy equation (2). In the neighbourhood of q we have the expansions

$$\chi_0 = \frac{1}{k} + a_0(x) + a_1(x)k + O(k^2), \quad \chi_1 = b_1(x)k + b_2(x)k^2 + O(k^3). \tag{8}$$

Functions χ_0, χ_1 have $2g$ simple poles $P_1(x), \dots, P_{2g}(x)$, and by Theorem 4

$$\chi_0(x,P) = \frac{-v_{i,0}(x)\gamma_i'(x)}{k-\gamma_i(x)} + d_{i,0}(x) + O(k-\gamma_i(x)), \tag{9}$$

$$\chi_1(x,P) = \frac{-\gamma_i'(x)}{k-\gamma_i(x)} + d_{i,1}(x) + O(k-\gamma_i(x)), \tag{10}$$

$$d_{i,0}(x) = v_{i,0}^2(x) + v_{i,0}(x)d_{i,1}(x) - v_{i,0}'(x). \tag{11}$$

Let Γ be the hyperelliptic spectral curve (1), $q = \infty \in \Gamma$, $k = \frac{1}{\sqrt{z}}$.

Let us find coefficients of the operator of order 4 corresponding to z, $L_4\psi = z\psi$.

Lemma 1 *The operator $L_4 = \partial_x^4 + f_2(x)\partial_x^2 + f_1(x)\partial_x + f_0(x)$ has the following coefficients:*

$$f_0 = a_0^2 - 2a_1 - 2b_1' - a_0'', \qquad f_1 = -2(b_1 + a_0'), \qquad f_2 = -2a_0.$$

Operator L_4 is self-adjoint if and only if $b_1 = 0$, herewith $L_4 = (\partial_x^2 + V(x))^2 + W(x)$, where $V(x) = -a_0(x)$, $W = -2a_1(x)$.

Proof. From (2) it follows that the fourth derivative of ψ is

$$\psi^{(4)} = (\chi_0^2 + \chi_1\chi_0' + \chi_0(\chi_1^2 + 2\chi_1') + \chi_0'')\psi + (\chi_1^3 + 2\chi_0' + \chi_1(2\chi_0 + 3\chi_1') + \chi_1'')\psi'.$$

With the help of (2) and the last equality we rewrite $L_4\psi = z\psi$ in the form

$$P_1\psi + P_2\psi' = z\psi,$$

where

$$P_1 = f_0 + f_2\chi_0 + \chi_0^2 + \chi_1\chi_0' + \chi_0(\chi_1^2 + 2\chi_1') + \chi_0'',$$
$$P_2 = f_1 + f_2\chi_1 + \chi_1^3 + 2\chi_0' + \chi_1(2\chi_0 + 3\chi_1') + \chi_1''.$$

This gives

$$P_1 = z = \frac{1}{k^2}, \qquad P_2 = 0. \tag{12}$$

From (8) we have

$$P_1 - \frac{1}{k^2} = \frac{f_2 + 2a_0}{k} + (f_0 + a_0(f_2 + a_0) + 2(a_1 + b_1') + a_0'') + O(k) = 0,$$
$$P_2 = (f_1 + 2(b_1 + a_0')) + O(k) = 0.$$

From here we find the coefficients of L_4.

Operator L_4 is self-adjoint if $f_1 = f_2'$, i.e. at $b_1 = 0$. Lemma 1 is proved.

If χ_1 satisfies (3) then $\chi_1 = \sum_{s>1} b_{2s}k^{2s}$, hence, by Lemma 1 L_4 is self-adjoint.

Let us prove the inverse part of Theorem 1. We assume that L_4 is self-adjoint

$$L_4 = L_4^* = \partial_x^4 + f_2(x)\partial_x^2 + f_2'(x)\partial_x + f_0(x).$$

If $\psi_1, \psi_2 \in Ker(L_4 - z)$, then

$\psi_1 L_4 \psi_2 - \psi_2 L_4 \psi_1 = \partial_x(\psi_1\psi_2''' - \psi_2\psi_1''' - (\psi_1'\psi_2'' - \psi_2'\psi_1'') + f_2(\psi_1\psi_2' - \psi_2\psi_1')) = 0.$

Hence, on the space $Ker(L_4 - z)$ the following skew-symmetric bilinear form

$$(.,.) : Ker(L_4 - z) \times Ker(L_4 - z) \to \mathbb{C},$$

$$(\psi_1, \psi_2) = \psi_1\psi_2''' - \psi_2\psi_1''' - (\psi_1'\psi_2'' - \psi_2'\psi_1'') + f_2(\psi_1\psi_2' - \psi_2\psi_1')$$

is defined. Let $\psi_1(x, P), \psi_2(x, P)$ satisfy the equation (2). Using

$$\psi_i''' = (\chi_0 + \chi_1^2 + \chi_1')\psi_i' + (\chi_0\chi_1 + \chi_0')\psi_i$$

we get

$$(\psi_1, \psi_2) = (\psi_1\psi_2' - \psi_2\psi_1')(f_2 + 2\chi_0 + \chi_2^2 + \chi_1').$$

Since ψ_1, ψ_2 satisfy the second order differential equation (2) we have,

$$(\psi_1, \psi_2) = e^{\int \chi_1(x,z,w)dx} g_1(z, w) \left(f_2(x) + 2\chi_0(x, z, w) + \chi_1^2(x, z, w) + \chi_1'(x, z, w)\right) = g_2(z, w),$$

where $g_1(z, w), g_2(z, w)$ are some functions on Γ. Let us represent χ_1 in the form

$$\chi_1(x, z, w) = G_1(x, z) + wG_2(x, z),$$

where G_1, G_2 are rational functions on Γ. Let

$$\tilde{G}_1(x, z) = \int G_1(x, z)dx, \qquad \tilde{G}_2(x, z) = \int G_2(x, z)dx,$$

then

$$e^{\tilde{G}_1(x,z)} \left(e^{\tilde{G}_2(x,z)}\right)^w \frac{g_1(z, w)}{g_2(z, w)} = \frac{1}{f_2 + 2\chi_0 + \chi_2^2 + \chi_1'}.$$

From the last identity it follows that for arbitrary $x = x_1, x = x_2$ the function

$$e^{\tilde{G}_1(x_1,z) - \tilde{G}_1(x_2,z)} \left(e^{\tilde{G}_2(x_1,z) - \tilde{G}_2(x_2,z)}\right)^w$$

is a rational function on Γ. This is possible only if

$$\tilde{G}_2(x_1, z) - \tilde{G}_2(x_2, z) = 0,$$

or equivalent $G_2 = 0$. Hence, $\chi_1 = G_1(x, z)$. This means that χ_1 is invariant under the involution σ. Thus, Theorem 1 is proved.

4. Proof of Theorem 2

Assume that χ_1 is invariant under σ, then by (8)–(10) we have

$$\chi_0 = \frac{H_1(x)}{z - \gamma_1(x)} + \cdots + \frac{H_g(x)}{z - \gamma_g(x)} + \frac{w(z)}{(z - \gamma_1(x)) \dots (z - \gamma_g(x))} + \kappa(x),$$

$$\chi_1(x, P) = -\frac{\gamma_1'(x)}{z - \gamma_1(x)} - \cdots - \frac{\gamma_g'(x)}{z - \gamma_g(x)},$$

where $H_i(x), \kappa(x)$ are some functions. In the neighbourhood of q the function χ_0 has the expansion

$$\chi_0 = \frac{1}{k} + \kappa + \left(\gamma_1 + \cdots + \gamma_g + \frac{c_{2g}}{2}\right) k + O(k^2).$$

Hence, by Lemma 1

$$V = -\kappa, \qquad W = -2(\gamma_1 + \cdots + \gamma_g) - c_{2g}. \tag{13}$$

Thus

$$\chi_0 = \frac{Q_1}{Q} + \frac{w}{Q} - V(x), \qquad \chi_1(x,P) = \frac{Q'}{Q}.$$

Let us substitute χ_0, χ_1 into (12). From $P_2 = 0$ we get $Q_1 = -\frac{Q''+s}{2}$, where s is a constant. From $P_1 = z$ we get

$$s^2 - 4sw + 4w^2 - 4(z-W)Q^2 + 4V(Q')^2 - (Q'')^2 + 2Q'Q^{(3)}$$
$$-2Q(2V'Q' + 4VQ'' + Q^{(4)}) = 0.$$

The last identity is possible only if $s = 0$ because Q is a polynomial in z. Theorem 2 is proved.

Let us differentiate (4) in x and divide the result by Q. We get the following equation.

Corollary 1 *The functions Q, W, V satisfy the equation*

$$Q^{(5)} + 4VQ^3 + 2Q'(2z - 2W - V'') + 6V'Q'' - 2QW' = 0.$$

Let us substitute $z = \gamma_j$ in (4). It gives

$$V(x) = \left(\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2}\right)|_{z=\gamma_j}.$$

We get $g-1$ equations on $\gamma_1(x), \dots, \gamma_g(x)$.

Corollary 2 *The functions $\gamma_1(x), \dots, \gamma_g(x)$ satisfy the equations*

$$\left(\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2}\right)|_{z=\gamma_j} = \left(\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2}\right)|_{z=\gamma_k}.$$

5. Proof of Theorem 3

Let

$$\chi_0 = -\frac{1}{2}\frac{Q''}{Q} + \frac{\sqrt{F_g(z)}}{Q} - (\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0), \tag{14}$$

$$\chi_1 = \frac{Q'}{Q}. \tag{15}$$

Let us consider the equations (4) where V, W are potentials of the operator $L_4^\sharp$

$$4F_g(z) = 4(z - g(g+1)\alpha_3 x)Q^2 - 4(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)(Q')^2 + (Q'')^2 - 2Q'Q^{(3)}$$

$$+ 2Q(2(3\alpha_3 x^2 + 2\alpha_2 x + \alpha_1)Q' + 4(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)Q'' + Q^{(4)}). \tag{16}$$

We prove that the nonlinear equation (16) has a polynomial solution $Q(x,z)$ of degree g in z and degree g in x for some polynomial $F_g(z)$. After that we prove that χ_0, χ_1 satisfy (11) for the curve $w^2 = F_g(z)$. The functions χ_0, χ_1 have required asymptotic (8) in $q = \infty$. From here it follows that $L_4^\sharp$ commutes with an operator of order $4g+2$ corresponding to the rational function w on Γ with the unique pole of order $2g+1$ at q.

Lemma 2 *Equation (16) has a solution of the form*

$$Q = (z - \gamma_1(x)) \dots (z - \gamma_g(x)), \tag{17}$$

for some polynomial $F_g(z)$ of degree $2g+1$.

Proof. Let us differentiate both sides of (16) with respect to x and divide the result by Q

$$Q^{(5)} + 4(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0) Q^{(3)} + 4(\alpha_2 - (g^2 + g - 3)\alpha_3 x + z) Q'$$

$$+ 6(3\alpha_3 x^2 + 2\alpha_2 x + \alpha_1) Q'' - 2g(g+1)\alpha_3 Q = 0. \tag{18}$$

We find a solution of (18) as a polynomial in x

$$Q = \delta_g x^g + \dots + \delta_1 x + \delta_0, \qquad \delta_i = \delta_i(z). \tag{19}$$

From (18) we have

$$\delta_s = \frac{(s+1)}{\alpha_3 (g-s)(s+g+1)(2s+1)} \big(2(\alpha_2 (s+1)^2 + z)\delta_{s+1} + \alpha_1 (s+2)(2s+3)\delta_{s+2}$$

$$+2\alpha_0 (s+2)(s+3)\delta_{s+3} + 1/2(s+2)(s+3)(s+4)(s+5)\delta_{s+5} \big), \tag{20}$$

where $0 \le s < g-1$, δ_g is a constant, and $\delta_s = 0$ at $s > g$. In particular

$$\delta_{g-1} = \frac{\delta_g (\alpha_2 g^2 + z)}{\alpha_3 (2g-1)}. \tag{21}$$

From (20) it follows that Q is a polynomial of degree g in z, and up to the multiplication by a constant, the polynomial Q has the form (17). The right-hand side of (16) has degree $2g+1$. Lemma 2 is proved.

Lemma 3 *The polynomial Q has no multiple root in z*

$$\gamma_i \neq \gamma_j \ \text{at}\ i \neq j.$$

Proof. Let us represent Q in the form

$$Q = Q_H + \tilde{Q},$$

where Q_H is a homogeneous polynomial in x, z

$$Q_H = \tilde{\delta}_g x^g + \tilde{\delta}_{g-1} x^{g-1} z + \tilde{\delta}_{g-2} x^{g-2} z^2 + \dots + \tilde{\delta}_0 z^g, \qquad \tilde{\delta}_0, \tilde{\delta}_g \neq 0$$

and $\deg \tilde{Q} < g$. Since $\tilde{\delta}_g \neq 0$, the polynomial Q has no constant roots (i.e. $\gamma_i \neq$ *const*).

Let us note that Q has no multiple roots of order higher than 2. Indeed, if $Q = (z - \gamma_i(x))^p \hat{Q},\ p > 2$, then from (16) $F_g(\gamma_i(x)) = 0$, but this is impossible.

If Q has multiple roots, then Q_H also has multiple roots. This follows from the following fact. The discriminant of Q is a polynomial $b_N x^N + b_{N-1} x^{N-1} + \dots + b_0$ in x. The discriminant of Q_H is $b_N x^N$, so if the discriminant of Q is equal to zero, then the discriminant of Q_H is also zero.

From (20) it follows that

$$\tilde{\delta}_s = \frac{2(s+1)\tilde{\delta}_{s+1}}{\alpha_3 (g-s)(s+g+1)(2s+1)}, \qquad 0 \le s \le g-1,$$

and that Q_H satisfies the equation

$$2\alpha_3x^3Q_H^{(3)}+2((3-g-g^2)\alpha_3x+z)Q'_H+9\alpha_3x^2Q''_H-g(g+1)\alpha_3Q_H=0.$$

Let us multiply this equation by Q_H and integrate in x. We get

$$\tilde F_g(z)+(g(g+1)\alpha_3x-z)Q_H^2+\alpha_3x^3(Q'_H)^2-\alpha_3x^2Q_H(3Q'_H+2xQ''_H))=0,$$

where $\tilde F_g(z)$ is a polynomial of degree $2g+1$ in z.

From the last equation it follows that if Q_H has multiple roots, then the polynomial $\tilde F_g(z)$ has the same roots. However, this is impossible, because all roots of $\tilde F_g(z)$ are constant, but Q_H has not constant roots. Lemma 3 is proved.

Lemma 4 *If $(\alpha_0,\dots,\alpha_3)\in U$, the curve $w^2=F_g(z)$ is nonsingular, where $U\subset\mathbb{C}^4$ is some Zariski open set.*

Proof. The idea of the proof is the following. We represent F_g in the form

$$F_g(z)=F_g^0(z)+\alpha_3F_g^1(z)+O(\alpha_3^2),$$

and prove that $F_g^0(z)+\alpha_3F_g^1(z)$ has not multiple roots. Therefore, $F_g(z)$ has not multiple roots for small α_3, and consequently for $(\alpha_0,\dots,\alpha_3)\in U$.

Let us consider (19)–(21). We put $\delta_g=\alpha_3^g$, then

$$\delta_{g-1}=\alpha_3^{g-1}\frac{\alpha_2g^2+z}{2g-1}.$$

Moreover, from (19) it follows that Q has the form

$$Q=\alpha_3^gx^g+\dots+\alpha_3^sx^s(p_s(z)+\alpha_3q_s(z)+O(\alpha_3^2))+\dots+(p_0(z)+\alpha_3q_0(z)+O(\alpha_3^2)). \tag{22}$$

Let us note that from (21) it follows that

$$p_g=1,\qquad p_{g-1}=\frac{\alpha_2g^2+z}{2g-1},\qquad q_g=0,\qquad q_{g-1}=0.$$

Let us substitute (22) into (16). We get

$$F_g(z)=p_0^2(z)z+\alpha_3p_0(z)(\alpha_1p_1(z)+2q_0(z)z)+O(\alpha_3^2),$$

so,

$$F_g^0(z)=p_0^2(z)z,\qquad F_g^1(z)=p_0(z)(\alpha_1p_1(z)+2q_0(z)z).$$

To prove Lemma 4 it is enough to prove that $p_0(z)z$ and $\alpha_1p_1(z)+2q_0(z)z$ have no common roots.

Let us find p_i and q_i. For this we again substitute (22) into (18) and find the coefficients at $\alpha_3^{i+1}x^i$ and $\alpha_3^{i+2}x^i$. These coefficients must be equal to zero. It gives us

$$p_i=\frac{2(i+1)(\alpha_2(i+1)^2+z)}{(2i+1)(g^2+g-i^2-i)}p_{i+1},\qquad 0\le i\le g-1, \tag{23}$$

$$q_i=\frac{2(i+1)(\alpha_2(i+1)^2+z)}{(g-i)(g+i+1)(2i+1)}q_{i+1}+\frac{\alpha_1(i+1)(i+2)(2i+3)}{(g-i)(g+i+1)(2i+1)}p_{i+2}, \tag{24}$$

where $0\le i\le g-2$. Hence

$$p_i(z)=(\alpha_2(i+1)^2+z)\dots(\alpha_2g^2+z)A_i,\qquad 0\le i\le g-1,$$

where A_i is a constant. Thus to prove that $p_0(z)z$ and $\alpha_1p_1(z)+2q_0(z)z$ have no common roots we should prove that $z=-\alpha_22^2,\dots,z=-\alpha_2g^2$ are not roots of

$q_0(z)$. Assume that $q_0(-\alpha_2 s^2)=0$ for some $s,\ 2\le s\le g$. From (23) it follows that $p_k(-\alpha_2 s^2)=0$ at $0\le k<s$, $p_k(-\alpha_2 s^2)\ne 0$ at $k\ge s$, and from (24) it follows that $q_k(-\alpha_2 s^2)=0$ at $0\le k\le s-2$.

First of all we consider the case $s=g$. If $i=g-2$, then (24) yields

$$q_{g-2}(z)=\frac{\alpha_1(g-1)g(2s-1)}{2(2g-1)(2g-3)}p_g(z).$$

Hence, if $q_0(-\alpha_2 g^2)=0$, then $q_{g-2}(-\alpha_2 g^2)=0$, but this is impossible, since $p_g=1$, so $s<g$.

Formulas (23), (24) at $i=s-2, i=s-1$ give us

$$q_{s-2}-\frac{2(s-1)(\alpha_2(s-1)^2+z)}{(g-s+2)(g+s-1)(2s-3)}\frac{2s(\alpha_2 s^2+z)q_s+\alpha_1 s(s+1)(2s+1)p_{s+1}}{(g-s+1)(g+s)(2s-1)}-$$

$$\frac{\alpha_1(s-1)s(2s-1)}{(g-s+2)(g+s-1)(2s-3)}\frac{2(s+1)(\alpha_2(s+1)^2+z)}{(2s+1)(g^2+g-s^2-s)}p_{s+1}=0.$$

Let z be $-\alpha_2 s^2$. After the simplification we have

$$g^2+g-3s^2=0.$$

This is impossible, hence $q_0(-\alpha_2 s^2)\ne 0$ and $F_g^0(z)+\alpha_3 F_g^2(z)$ has no multiple roots. Lemma 4 is proved.

Functions χ_0,χ_1 are rational functions on the curve $w^2=F_g(z)$. Let $k=\frac{1}{\sqrt{z}}$ be a local parameter near $q=\infty$. Functions χ_0,χ_1 have asymptotic (3). By Lemma, 3 χ_0 and χ_1 have simple poles $P_i^{\pm}=(\gamma_i,\pm\sqrt{F_g(\gamma_i)})$. Let us choose in the neighbourhood of $P_i^{\pm}$ the local parameter $z-\gamma_i(x)$.

Lemma 5 *Functions χ_0,χ_1 satisfy the equation (11).*

Proof. From (15) we have

$$\chi_1(x,P)=\frac{-\gamma_i'(x)}{z-\gamma_i(x)}+d_{i,1}(x)+O(z-\gamma_i(x))$$

for some $d_{i,1}(x)$. Function χ_1 has simple poles at $\gamma_i(x)$, thus

$$\chi_0(x,P)=\frac{-v_{i,0}(x)\gamma_i'(x)}{z-\gamma_i(x)}+d_{i,0}(x)+O(z-\gamma_i(x)),$$

for some $v_{i,0}(x),d_{i,0}(x)$. By our construction χ_0, χ_1 satisfy (12). Let us substitute χ_0, χ_1 in (12). We get

$$\frac{(v_{i,0}^2(x)-d_{i,0}(x)+d_{i,1}(x)v_i(x)-v_i'(x))(\gamma_i'(x))^2}{(z-\gamma_i(x))^2}+O\left(\frac{1}{z-\gamma_i(x)}\right)=0.$$

Hence $d_{i,0}(x),d_{i,1}(x),v_{i,0}(x)$ satisfy (11). Lemma 5 and Theorem 3 are proved.

Operator $L_{4g+2}^{\sharp}$ commuting with $L_4^{\sharp}$ can be found from $L_4^{\sharp}L_{4g+2}^{\sharp}=L_{4g+2}^{\sharp}L_4^{\sharp}$. For the simplicity of the formulas we restrict ourselves to the case $\alpha_1=\alpha_2=0,\alpha_3=1$. Let us introduce the notations: $H=\partial_x^2+x^3+\alpha_0,\ \langle A,B\rangle=AB+BA$.

Examples.

a) $g = 2$:

$$L^{\sharp}_{10} = H^5 + \frac{15}{2}\langle x, H^3\rangle + 45\langle x^2, H\rangle,$$

$$F_2(z) = z^5 + 27\alpha_0 z^2 + 81.$$

b) $g = 3$:

$$L^{\sharp}_{14} = H^7 + 21\langle x, H^5\rangle + \frac{945}{2}\langle x^2, H^3\rangle - 5418H^2 + \frac{45}{2}\langle 113\alpha_0 + 287x^3, H\rangle - 486x,$$

$$F_3 = z^7 + 594\alpha_0 z^4 - 2025z^2 + 91125\alpha_0^2 z.$$

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